Synchronization of chaos and hyperchaos using linear and nonlinear feedback functions

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Using the method of variable feedback, synchronization of chaotic and hyperchaotic systems is presented. The robustness of the method based on the flexibility of choices of feedback functions is exemplified. Linear and nonlinear feedback functions and their linear superpositions are used for synchronization. Calculations with model systems indicate that functions constructed by linear superposition of feedback functions that independently synchronize a system are more efficient in achieving synchronization than the functions from which they are made. We have not noticed any difference in the technique of synchronization based on the number of positive Lyapunov exponents. [S1063-651X(97)09605-0]

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I. INTRODUCTION

Quite recently, synchronization of chaos and hyperchaos has received increasing attention from foundational as well as applied points of view [1–9]. Systems with more than one positive Lyapunov exponent are called hyperchaotic [10]. For bounded flows, chaos and hyperchaos can exist only if the minimum dimensions of dynamical systems are 3 and 4, respectively. Examples of systems having chaos and hyperchaos can be found in systems [1–14] such as arrays of Josephson junctions, coupled electric oscillators, artificial neural networks, arrays of chaotic systems, multimode lasers, and coupled map lattices. The importance of a proper understanding of control and synchronization of chaotic systems originates from the fact that in the classical world chaos is ubiquitous. Synchronization of chaos has been studied in the light of technical applications such as secure communications [1,2,4–6]. It is very likely that control and synchronization of chaos and hyperchaos play important roles in the workings of biological and artificial neural networks. Although some progress has been made, an all-embracing unified method for synchronization of chaos and hyperchaos is missing. Kocarev and Parlitz [4,5] and Pen *et al.* [6] have recently studied synchronization of high-dimensional systems in which the transmitted signal is hyperchaotic. It is believed that communications via hyperchaotic signals are more secure and the procedure relates to the design of communication schemes. Synchronization of hyperchaotic systems has therefore become a new area of active research. One way to construct hyperchaotic systems is to couple low-dimensional chaotic (hyperchaotic) systems [4,5,11]. A system built by coupling simpler chaotic (hyperchaotic) systems may behave [5] as a single chaotic or hyperchaotic system with new and rich dynamic behavior or it may behave as clusters of chaotic (hyperchaotic) systems or it may even exhibit the behaviors of an assembly of individual chaotic (hyperchaotic) systems. Quite often, a complex physical or biological system can be described in terms of the behaviors of simpler chaotic (hyperchaotic) systems through a mechanism dubbed *dynamic dissipation* [7,11]. In this work, we study synchronization of chaos and hyperchaos using the method of variable feedback control. So far, this method has been used primarily with linear feedback functions. We focus on linear and and nonlinear feedback functions and their superposition. Since feedback functions are not unique, one may ask questions such as the following: (i) If two vector feedback functions $\mathbf{G}_\nu$ and $\mathbf{G}_\mu$ synchronize a chaotic (hyperchaotic) system with its replica, will their superposition $\mathbf{G}=\mathbf{G}_\nu+\mathbf{G}_\mu$ synchronize the same system? (ii) If $\mathbf{G}$ synchronizes the system, how does its synchronization efficiency compare with those of $\mathbf{G}_\nu$ and $\mathbf{G}_\mu$? The questions are relevant, but their answers are not obvious. By considering linear-linear, linear-nonlinear, and nonlinear-nonlinear combinations of feedback functions, we provide numerical evidence that $\mathbf{G}=\mathbf{G}_1+\mathbf{G}_2+\cdots+\mathbf{G}_n$ synchronizes a chaotic (hyperchaotic) system if each of the $\mathbf{G}_i, i=1,2,\ldots,n$ synchronizes the system independently and $\mathbf{G}$ is not less efficient than any of the $\mathbf{G}_i$, $i=1,2,\ldots,n$. Our observation is valid for at least a class of systems some members of which we have studied numerically.

II. VARIABLE FEEDBACK CONTROL

In this section, a brief description of the method of variable feedback control is given. Let us consider two n-dimensional autonomous dynamical systems described by the differential equations

$$\dot{\mathbf{X}}=\mathbf{F}(\mathbf{X}), \tag{1}$$

$$\dot{\mathbf{Y}}=\mathbf{F}(\mathbf{Y})+\mathbf{G}(\mathbf{X},\mathbf{Y}). \tag{2}$$

Here the state vectors $\mathbf{X},\mathbf{Y}\in\mathbf{V}(\mathbb{R}^n)$ are n-dimensional vectors with components $X_1,X_2,\ldots,X_n$ and $Y_1,Y_2,\ldots,Y_n$, respectively. The vector functions $\mathbf{F}$ and $\mathbf{G}$ have the components $F_1,F_2,\ldots,F_n$ and $G_1,G_2,\ldots,G_n$, respectively. The dynamical system of Eq. (1) is variably called the driver, master, or sender system, while the system of Eq. (2) is called the response, slave, or receiver system. The function $\mathbf{F}(\mathbf{Y})$ is a replica of the function $\mathbf{F}(\mathbf{X})$ and $\mathbf{G}(\mathbf{X},\mathbf{Y})$ is called

TABLE I. Examples of feedback functions for synchronizing the Rössler system. The response system of equations (12)–(14) will synchronize with the driver system of equations (9)–(11) when the feedback functions G_1, G_2, and G_3 from any row are used. The synchronization time τ provides a measure of the efficiency of the corresponding set of feedback functions.

G_1	G_2	G_3	τ
$Y_2[(Y_2-X_2)-(Y_1-X_1)]$	$(Y_1-X_1)-(Y_2-X_2)$	0	43
$3[(Y_2-X_2)+(Y_3-X_3)]$			
$-(Y_1Y_2-X_1X_2)$	0	0	150
	$-(Y_1-X_1)-(Y_2-X_2)$		
0	$-(Y_3-X_3)$	0	23
0	0	$-0.45(Y_3-X_3)$	78

the feedback function. The dynamics of the **X** and **Y** systems are started from different initial conditions $[\mathbf{Y}(t=0)\neq\mathbf{X}(t=0)]$. Synchronization between the driver and response systems is said to be achieved [4,5] if the dynamical system describing the time evolution of the difference $\mathbf{e}=\mathbf{Y}-\mathbf{X}$,

$$\begin{aligned}\dot{\mathbf{e}}&=\mathbf{F}(\mathbf{Y})+\mathbf{G}(\mathbf{X},\mathbf{Y})-\mathbf{F}(\mathbf{X})\\&=\mathbf{F}(\mathbf{X}+\mathbf{e})+\mathbf{G}(\mathbf{X},\mathbf{X}+\mathbf{e})-\mathbf{F}(\mathbf{X}),\end{aligned}\tag{3}$$

has a stable fixed point at $\mathbf{e}=\mathbf{0}$. Another way of saying this is that we achieve synchronization if $\|\mathbf{Y}-\mathbf{X}\|\to 0$ and $\mathbf{G}(\mathbf{X},\mathbf{Y})\to 0$ as $t\to\infty$. For the response system to synchronize, it is necessary that all of its Lyapunov exponents (conditional Lyapunov exponents) are negative [1,2]. Let $\delta\mathbf{Y}$ be a small change in **Y**. Then in the linear approximation we have the variational equation

$$\frac{d\,\delta\mathbf{Y}}{dt}=\nabla_{\mathbf{Y}}[\mathbf{F}(\mathbf{Y})+\mathbf{G}(\mathbf{X},\mathbf{Y})]\cdot\delta\mathbf{Y}.\tag{4}$$

Here $\nabla_{\mathbf{Y}}$ represents the gradient with respect to **Y**. For computing the Lyapunov exponents of the response system, Eqs. (1), (2), and (4) are solved simultaneously. The above synchronization conditions can be satisfied by a very large number of linear [6,15–20] and nonlinear functional forms of $\mathbf{G}(\mathbf{X},\mathbf{Y})$. Pyragas [15] has used $G_i=k(Y_i-X_i)$, where k is suitably chosen. Pen *et al.* [6] have used $\mathbf{G}(\mathbf{X},\mathbf{Y})=\mathbf{A}(\mathbf{Y}-\mathbf{X})$, where **A** is a matrix. These forms of $\mathbf{G}(\mathbf{X},\mathbf{Y})$ provide linear feedback. However, $\mathbf{G}(\mathbf{X},\mathbf{Y})$ does not have to be restricted to linear forms and we show below that linear as well as nonlinear forms of the feedback function are effective in synchronizing chaotic and hyperchaotic systems. Recently, Parlitz *et al*. have used an active-passive decomposition scheme for synchronization [4,5]. This scheme involves decomposing the dynamical systems as

$$\dot{\mathbf{X}}=\mathbf{F}(\mathbf{X},\mathbf{s}),\tag{5}$$

$$\dot{\mathbf{Y}}=\mathbf{F}(\mathbf{Y},\mathbf{s}),\tag{6}$$

$$\mathbf{s}=\mathbf{h}(\mathbf{X}),\tag{7}$$

$$\dot{\mathbf{s}}=\mathbf{h}(\mathbf{X},\mathbf{s}).\tag{8}$$

Mathematically, this decomposition scheme is included in our approach if we choose $\mathbf{G}(\mathbf{X},\mathbf{Y})=\mathbf{F}(\mathbf{Y},\mathbf{s})-\mathbf{F}(\mathbf{Y})$. Parlitz *et al.* [5] have shown the mathematical equivalence between their decomposition scheme and the approaches of Pecora and Carroll [1,2]. Therefore, the feedback scheme of Eqs. (1) and (2) not only includes the approaches just mentioned but also has the flexibility of introducing new feedback functions and thus giving robustness to the method of feedback control. The evidence is that a dynamical system can be synchronized by a variety of choices of the feedback function $\mathbf{G}(\mathbf{X},\mathbf{Y})$. The important question about how to find all of them from a single general method is still unanswered. Because of the robustness of the method, it does not seem to be hard to find some feedback functions for a given system by trial and error.

III. MODEL DYNAMICAL SYSTEMS

In this section we present some of the dynamical systems that we have considered.

A. Chaotic systems

1. The Rössler system

The well-known Rössler attractor (Lyapunov spectrum $0.11,0.00,-3.21$) [10] and its response systems are described by

$$\dot{X}_1=F_1(\mathbf{X})=2+X_1(X_2-4),\tag{9}$$

$$\dot{X}_2=F_2(\mathbf{X})=-X_1-X_3,\tag{10}$$

$$\dot{X}_3=F_3(\mathbf{X})=X_2+0.45X_3,\tag{11}$$

$$\dot{Y}_1=2+Y_1(Y_2-4)+G_1(\mathbf{X},\mathbf{Y}),\tag{12}$$

$$\dot{Y}_2=-Y_1-Y_3+G_2(\mathbf{X},\mathbf{Y}),\tag{13}$$

$$\dot{Y}_3=Y_2+0.45Y_3+G_3(\mathbf{X},\mathbf{Y}).\tag{14}$$

Recently, Parlitz *et al.* [5] have considered examples of active-passive decompositions of this system. These sample decompositions correspond to a set of feedback functions $G_i(\mathbf{X},\mathbf{Y})$, $i=1,2,3$, which are given in Table I. It can be seen from the table that linear as well as linear-nonlinear feedback injected at one or more than one site can synchro-

TABLE II. Examples of linear and nonlinear feedback functions for synchronizing the Lorenz system. Any set of the feedback functions chosen from any column will synchronize the Lorenz system with its replica. The approximate lower bounds l of the coupling parameters K_i^μ are given. τ (see the Rössler system for the definition) provides a measure of the efficiency of the corresponding choice of G_1, G_2, and G_3.

$-G_1(G_2=G_3=0)$	$-G_2(G_1=G_3=0)$	$-G_3(G_1=G_2=0)$
$K_1^1(Y_1-X_1)$	$K_2^1(Y_1-X_1)$	$K_3^1(Y_3-X_3)$
$8\sim l<K_1^1$	$6.5\sim l<K_2^1$	$2<K_3^1$
$K_1^1=9,\ \tau=29$	$K_2^1=9,\ \tau=7$	$K_3^1=9,\ \tau=4$
$K_1^2(Y_2-X_2)$	$K_2^2(Y_2-X_2)$	
$5\sim l<K_1^2$	$3\sim l<K_2^2$	
$K_1^2=9,\ \tau=8$	$K_2^2=9,\ \tau=4$	
$K_1^3\sin(Y_1-X_1)$	$K_2^3\sin(Y_1-X_1)$	
$8\sim l<K_1^3$	$6.5\sim l<K_2^3$	
$K_1^3=9,\ \tau=29$	$K_2^3=9,\ \tau=7$	
$K_1^4\sin(Y_2-X_2)$	$K_2^4\sin(Y_2-X_2)$	$K_3^2\sin(Y_3-X_3)$
$5\sim l<K_1^4$	$3\sim l<K_2^4$	$2\sim l<K_3^2$
$K_1^4=9,\ \tau=8$	$K_2^4=9,\ \tau=4$	$K_3^2=9,\ \tau=4$
$K_1^5\tanh(Y_1-X_1)$	$K_2^5\tanh(Y_1-X_1)$	
$8\sim l<K_1^5$	$7\sim l<K_2^5$	
$K_1^5=9,\ \tau=21$	$K_2^5=9,\ \tau=7$	
$K_1^6\tanh(Y_2-X_2)$	$K_2^6\tanh(Y_2-X_2)$	$K_3^3\tanh(Y_3-X_3)$
$5\sim l<K_1^6$	$3\sim l<K_2^6$	$2\sim l<K_3^3$
$K_1^6=9,\ \tau=8$	$K_2^6=9,\ \tau=4$	$K_3^3=9,\ \tau=4$

TABLE III. Examples of feedback functions for synchronizing the Van der Pol–Duffing oscillator. The approximate lower bounds l of the coupling parameters K_i^μ are given. See the Rössler system for the definition of τ.

$-G_1(G_2=G_3=0)$	$-G_2(G_1=G_3=0)$	$-G_3(G_1=G_2=0)$
$K_1^1(Y_1-X_1)$	$K_2^1(Y_1-X_1)$	$K_3^1(Y_3-X_3)$
$74\sim l<K_1^1$	$0.84\sim l<K_2^1$	$3<K_3^1<80$
$K_1^1=80,\ \tau=248$	$K_2^1=0.9,\ \tau=49$	$K_3^1=3.5,\ \tau=15$
$K_1^2(Y_2-X_2)$	$K_2^2(Y_2-X_2)$	
$83\sim l<K_1^2$	$3.5\sim l<K_2^2$	
$K_1^2=85,\ \tau=236$	$K_2^2=4,\ \tau=16$	
$K_1^3\sin(Y_1-X_1)$	$K_2^3\sin(Y_1-X_1)$	
$75\sim l<K_1^3$	$0.85\sim l<K_2^3$	
$K_1^3=80,\ \tau=268$	$K_2^3=0.9,\ \tau=76$	
$K_1^4\sin(Y_2-X_2)$	$K_2^4\sin(Y_2-X_2)$	
$84\sim l<K_1^4$	$3.5\sim l<K_2^4$	
$K_1^4=85,\ \tau=140$	$K_2^4=4,\ \tau=17$	
$K_1^5\tanh(Y_1-X_1)$	$K_2^5\tanh(Y_1-X_1)$	
$80\sim l<K_1^5$	$0.84\sim l<K_2^5$	
$K_1^5=81,\ \tau=300$	$K_2^5=0.9,\ \tau=113$	
$K_1^6\tanh(Y_2-X_2)$	$K_2^6\tanh(Y_2-X_2)$	$K_3^6\tanh(Y_3-X_3)$
$84\sim l<K_1^6$	$3\sim l<K_2^6$	$5\sim l<K_3^6$
$K_1^6=85,\ \tau=153$	$K_2^6=4,\ \tau=17$	$K_3^6=6,\ \tau=20$

TABLE IV. Examples of feedback functions for synchronizing the Lorenz–Van der Pol–Duffing oscillator. Any set of the G_i, $i=1,2,\ldots,6$ from the categories A, B, C, and D will synchronize this hyperchaotic system. For the definition of τ see the Rössler attractor.

Category	Function
A	$G_2=G_3=G_5=G_6=0$
	$G_1=-K(Y_1-X_1),\ G_4=-K(Y_4-X_4),\ K>70,$
	$\tau=93$ when $K=100$
	$G_1=-K\sin(Y_1-X_1),\ G_4=-K\sin(Y_4-X_4),\ K>70,$
	$\tau=94$ when $K=100$
	$G_1=-K\tanh(Y_1-X_1),\ G_4=-K\tanh(Y_4-X_4),\ K>70,$
	$\tau=92$ when $K=100$
	$G_1=-K(Y_1-X_1),\ G_4=-K\sin[0.25(Y_4-X_4)],$
	$\tau=93$ when $K=100$
B	$G_2=G_3=G_4=G_6=0$
	$G_1=-K(Y_1-X_1),\ G_5=-K(Y_5-X_5),\ K>70,$
	$\tau=96$ when $K=100$
	$G_1=-K\sin(Y_1-X_1),\ G_5=-K\sin(Y_5-X_5),\ K>70,$
	$\tau=96$ when $K=100$
	$G_1=-K\tanh(Y_1-X_1),\ G_5=-K\tanh(Y_5-X_5),\ K>70,$
	$\tau=97$ when $K=100$
C	$G_1=G_3=G_5=G_6=0$
	$G_2=-K(Y_2-X_2),\ G_4=-K(Y_4-X_4),\ K>10,$
	$\tau=13$ when $K=15$
	$G_2=-K\sin(Y_2-X_2),\ G_4=-K\sin(Y_4-X_4),\ K>8,$
	$\tau=13$ when $K=15$
	$G_2=-K\tanh(Y_2-X_2),\ G_4=-K\tanh(Y_4-X_4),\ K>8,$
	$\tau=13$ when $K=15$
	$G_2=-K\tanh[5(Y_2-X_2)],\ G_4=-K\tanh[5(Y_4-X_4)],$
	$\tau=11.5$ when $K=10$
D	$G_1=G_3=G_4=G_6=0$
	$G_2=-K(Y_2-X_2),\ G_5=-K(Y_5-X_5),\ K>3,$
	$\tau=27$ when $K=5$
	$G_2=-K\sin(Y_2-X_2),\ G_5=-K\sin(Y_5-X_5),\ K>3,$
	$\tau=29$ when $K=5$
	$G_2=-K\tanh(Y_2-X_2),\ G_5=-K\tanh(Y_5-X_5),\ K>3,$
	$\tau=33$ when $K=5$
	$G_2=-K\tanh[5(Y_1-X_1)],\ G_5=-K\tanh[5(Y_5-X_5)],$
	$\tau=2.5$ when $K=10$
	$G_2=-K\sin[15(Y_2-X_2)],\ G_5=-K\tanh[5(Y_4-X_4)],$
	$\tau=4.5$ when $K=5$

nize the systems. In addition to the feedback functions, Table I also contains the "synchronization time" τ, which is taken here as the time required for $d=\sqrt{[\Sigma_i(Y_i-X_i)]^2}$ to reduce to 1×10^{-10}. τ is used here only for studying relative efficiencies of different feedback functions. As for τ for these sample feedback functions, the linear-nonlinear feedback functions do not have an advantage over the linear feedback functions. But for our present work we are interested in knowing that a variety of feedback functions exists. In Sec. IV, we will linearly superpose these feedback functions and study the resulting function from the point of view of synchronization.

2. The Lorenz system

In the Lorenz system

$$\dot{X}_1=10(-X_1+X_2), \tag{15}$$

TABLE V. Examples of superpositions of feedback functions for the Rössler attractor.

G_1	G_2	G_3	τ
$Y_2[(Y_2-X_2)-(Y_1-X_1)]$	$(Y_1-X_1)-(Y_2-X_2)$	$-0.45(Y_3-X_3)$	23
$Y_2[(Y_2-X_2)-(Y_1-X_1)]$ $+3[(Y_2-X_2)+(Y_3-X_3)]$ $-(Y_1Y_2-X_1X_2)$	$(Y_1-X_1)-(Y_2-X_2)$ $-(Y_1-X_1)-(Y_2-X_2)$ $-(Y_3-X_3)$	$-0.45(Y_3-X_3)$	8
$3[(Y_2-X_2)+(Y_3-X_3)]$ $-(Y_1Y_2-X_1X_2)$	$-(Y_1-X_1)-(Y_2-X_2)$ $-(Y_3-X_3)$	0	10
0	$-(Y_1-X_1)-(Y_2-X_2)$ $-(Y_3-X_3)$	$-0.45(Y_3-X_3)$	15
$Y_2[(Y_2-X_2)-(Y_1-X_1)]$ $+3[(Y_2-X_2)+(Y_3-X_3)]$ $-(Y_1Y_2-X_1X_2)$	$(Y_1-X_1)-(Y_2-X_2)$ $-(Y_1-X_1)-(Y_2-X_2)$ $-(Y_3-X_3)$	0	16

$$\dot{X}_2=-X_1X_3+28X_1-X_2, \tag{16}$$

$$\dot{X}_3=X_1X_2-\frac{8}{3}X_3. \tag{17}$$

The Lyapunov exponents of this well-studied system are 0.91, 0.00, and -14.57 and the **Y** system is written the same way as in the case of the Rössler system. Recently, Malescio [13] has studied synchronization of the Lorenz system using $\dot{X}_1=10(X_1-X_2)$. This difference in the sign of F_1 does affect the choice of some synchronization functions. Examples of linear and nonlinear feedback functions that synchronize this system are given in Table II.

3. The Van der Pol–Duffing oscillator

In the Van der Pol–Duffing system

$$\dot{X}_1=-100(X_1^3-0.35X_1-X_2), \tag{18}$$

$$\dot{X}_2=X_1-X_2-X_3, \tag{19}$$

$$\dot{X}_3=450X_3. \tag{20}$$

This chaotic system (Lyapunov exponents 1.29, 0.00, and -49.41) has been studied [11,12] in connection with signal transmission by chaos synchronization. We provide in Table III examples of linear and nonlinear feedback functions and the corresponding τ for this system.

B. Hyperchaotic system: The Lorenz–Van der Pol–Duffing system

For a hyperchaotic system, we consider the Lorenz–Van der Pol–Duffing (LVPD) system, which is obtained by coupling the Van der Pol–Duffing [11] and Lorenz systems as

$$\dot{X}_1=-100(X_1^3-0.35X_1-X_2), \tag{21}$$

$$\dot{X}_2=X_1-X_2-X_3, \tag{22}$$

$$\dot{X}_3=450X_2+0.1X_4, \tag{23}$$

$$\dot{X}_4=10(-X_4+X_5), \tag{24}$$

$$\dot{X}_5=-X_4X_6+28X_4-X_5, \tag{25}$$

$$\dot{X}_6=X_4X_5-\frac{8}{3}X_6. \tag{26}$$

The Lyapunov spectrum for this system is 1.14, 0.91, 0.00, -0.02, -14.57, and -48.77. Table IV contains examples of linear and nonlinear feedback functions for synchronizing this hyperchaotic system. Although the LVPD system is formed by coupling the Lorenz and Van der Pol–Duffing systems, not all combinations of the feedback functions that synchronize the Van der Pol–Duffing and the Lorenz systems separately are suitable for synchronizing the LVPD system.

IV. SUPERPOSITION OF FEEDBACK FUNCTIONS

Having examples of linear and nonlinear feedback functions for chaotic and hyperchaotic systems (Tables I–IV), we examine their behaviors in superposition. First consider the Rössler system. From the feedback functions given in Table I, we construct (by linear superposition) the functions G_1, G_2, and G_3, which are given in Table V. Our numerical results show (see Table V) that these superposed functions are the new feedback functions (constructed from the old ones). By feedback functions we mean functions that yield stable fixed point at $\mathbf{e}=\mathbf{0}$ for the time evolution of the difference $\mathbf{e}=\mathbf{Y}-\mathbf{X}$ [see Eq. (3)]. It is interesting to note that the synchronization time τ for any of the superposed feedback functions is shorter than that for any of its components. This behavior of τ is observed in all the chaotic and hyperchaotic systems that we have studied. For example, consider the Van der Pol–Duffing oscillator. Column 1 of Table III shows that the minimum τ of the six choices of G_1 is 153. If we superpose the six functions we obtain a new G_1 given by

$$\begin{aligned}G_1=&80(Y_1-X_1)+85(Y_2-X_2)+80\sin(Y_1-X_1)\\&+85\sin(Y_2-X_2)+81\tanh(Y_1-X_1)\\&+85\tanh(Y_2-X_2).\end{aligned} \tag{27}$$

The synchronization time for this new G_1 ($G_2=G_3=0$) is $\tau=18$, which is shorter than the minimum $\tau=153$ for its components. For another example, let us consider the hyper-

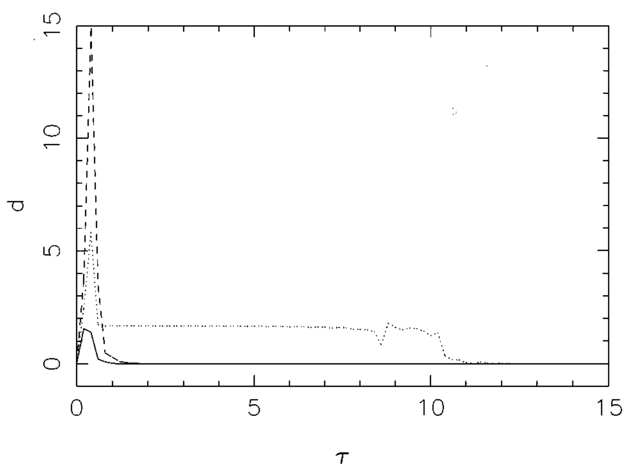

FIG. 1. Comparison of the synchronization time τ of three feedback functions for the LVPD system. Here τ is taken as the time required for $d=\sqrt{[\Sigma_i(Y_i-X_i)]^2}$ to reduce to 1×10^{-10}. For all curves in the figure, $G_1=G_3=G_4=G_6=0$. Curves $\cdots$, $---$, and —— respectively represent the linear feedback functions $G_2=-10(Y_2-X_2)$ and $G_5=-10(Y_5-X_5)$, the nonlinear feedback functions $G_2=-10\tanh[5(Y_1-X_1)]$ and $G_5=-10\tanh[5(Y_5-X_5)]$, and the superposed feedback functions $G_2=-10(Y_2-X_2)-10\tanh[5(Y_1-X_1)]$ and $G_5=-10(Y_5-X_5)-10\tanh[5(Y_5-X_5)]$. It can be seen that the superposed feedback function is more efficient than its components.

chaotic LVPD system. From groups A and C in Table IV, let us consider the two sets of feedback functions $G_1=-K(Y_1-X_1)$, $G_2=0$, $G_3=0$, $G_4=-K(Y_4-X_4)$, $G_5=G_6=0$ and $G_1=0$, $G_2=-K(Y_2-X_2)$, $G_3=0$, $G_4=-K(Y_4-X_4)$, $G_5=G_6=0$. For the first set $K>70$ and $\tau=93$ when $K=100$ while for the second set $K>10$ and $\tau=13$ when $K=15$. If we superpose these two sets we find $G_1=-K(Y_1-X_1)$, $G_2=-K(Y_2-X_2)$, $G_3=0$, $G_4=-K(Y_4-X_4)$, $G_5=G_6=0$. This new set of functions synchronizes the hyperchaotic system for $K>10$ and $\tau=12$ when $K=15$. This τ is shorter than the $\tau=13$ of the second set. Another point to note here is that the superposed feedback function synchronizes the hyperchaotic system for $K>10$, although one of its components (the first set) synchronizes only when $K>70$. Thus the superposition of feedback functions has affected the synchronization time as well as the coupling constant. We obtained similar results in our study of the hyperchaotic Chua circuit [21]. Figure 1 shows a comparison of synchronization efficiencies of (i) a linear feedback function, (ii) a nonlinear feedback function, and (iii) a feedback function obtained by linear superposition of the functions of (i) and (ii). It can be seen that the feedback function of (iii) is the fastest compared to the those of (i) and (ii).

What we have seen here is that superpositions of known feedback functions give new feedback functions for synchronizing chaotic and hyperchaotic systems. There may be one objection to obtaining feedback functions by superpositions. Consider, for example, Table V of the Rössler system. This table shows that some superpositions imply injecting the feedback to more than the minimum number of sites necessary for synchronizing this system. From an experimental point of view this may not be very appealing. However, beyond our current experimental limitations and interests, there are situations where the main concern regarding synchronization may be different from the requirement of the minimum number of such sites. For example, it is quite possible that the criteria for synchronization in biological neural networks are different from those for secure communications.

V. DISCUSSION AND CONCLUSIONS

In this work we have studied synchronization of chaotic and hyperchaotic systems using linear and nonlinear feedback functions and their superpositions. For a model hyperchaotic system, we have introduced a six-dimensional system by coupling the Lorenz and Van der Pol–Duffing systems. This hyperchaotic system has two positive Lyapunov exponents and we have shown that it can be synchronized by applying linear and nonlinear feedback. Comparing the efficiencies (values of τ) of linear and nonlinear feedback functions that we have considered, we do not see any real advantage of one form over the other. However, we have not found all the nonlinear feedback functions for our systems and hence we are unable to make a precise judgement about the relative merits of linear and nonlinear feedback functions. Further work needs to be done. As for synchronizing chaotic versus hyperchaotic systems, we did not need different approaches based on the number of positive Lyapunov exponents. Since a variety of feedback functions can synchronize a given chaotic (hyperchaotic) system, we have studied their superpositions from the point of view of synchronizing the same system. Our numerical results for the cases that we have studied show that functions obtained by superposition of known feedback functions do synchronize the same system and they are not less efficient than their components. We have also shown that the superposed feedback functions are effective in bringing synchronization with parameter values that are not suitable for some of its components.

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